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A Mathematical Model for Integrating Product of Two Functions

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Abstract

Integration by parts is a well-known method of integrating product of two functions. If the integrand involves a polynomial $f(x)$ of degree n and another function $g(x)$ that can be integrated at least (n+1) times, then the solution to $\int f(x)g(x)dx$ is guaranteed after n routine applications of integration by parts method. This paper presents a mathematical model for integrating product of two functions without going through the routine application of integration by parts at each stage of integration, thus saving a lot of computational time. Some examples were considered to illustrate the effectiveness of the model. The model is found to be appropriate as it gives the same result with the well-known integration by parts method.

Keywords: Integrand; Mathematical model; Integration by parts; Differentiable functions

1. Introduction

Integration by parts is a method used for integrating product of two functions particularly when either function is not a derivative of the other (Stroud & Dexter 2007).

If $f(x)$ and $g(x)$ are two differentiable functions of x, then

$$\frac{d}{dx}(f.g) = f\frac{dg}{dx} + g\frac{df}{dx} \quad (1)$$

(1) is the formula for the derivative of product of two functions as we can see in Bhattacharyya (2009), Matthew & Alabi (2008), Dass (2009), Bajpai et al. (1981), Richmond (1972) and Oke (2003) , where $f = f(x)\ and\ g = g(x)$.

Rearranging equation (1) we have:

$$f\frac{dg}{dx} = \frac{d}{dx}f.g - g\frac{df}{dx} \quad (2)$$

Integrating equation (2) with respect to x, we have:

$$\int fdg = f.g - \int gdf \quad (3)$$

(3) is the formula for integration by parts as we can see in Kreyszig (1987), Thomas JR. & Finney (1984), Daniel (2000), Gupta (2009), Grossman (1985), and Goldstein et al. (2007).

Let us assume that $f(x)$ is a polynomial of degree n and let $g(x)$ be a function of x that can be integrated at least (n+1) times. To evaluate $\int f(x)g(x)dx$ using the integration by parts method, we will need n applications of the formula in (3) above before we can get a solution. In this paper, we derived a mathematical model for integrating product of two functions. In applying the model, we don't need to go through the routine application of integration by parts each time we are integrating product of two functions. We only need to substitute the derivatives of $f(x)$ and the integrals of $g(x)$ in the newly derived model.

2. Materials and Methods

Let $f(x)$ and $g(x)$ be two functions of x, where $f(x)$ is a polynomial of degree n and $g(x)$ is a function of x that can be integrated at least (n+1) times. Then:

$$\int f(x)g(x)dx = f(x)\int g(x)dx - \int(f^1(x)\int g(x)dx)dx \tag{4}$$

$$= f(x)\int g(x)dx - f^1(x)\iint g(x)(dx)^2 + \int(f^2(x)\iint g(x)(dx)^2)\,dx \tag{5}$$

$$= f(x)\int g(x)dx - f^1(x)\iint g(x)(dx)^2 + f^2(x)\iiint g(x)(dx)^3$$
$$- \int(f^3(x)\iiint g(x)(dx)^3)dx \tag{6}$$

$$= f(x)\int g(x)dx - f^1(x)\iint g(x)(dx)^2 + f^2(x)\iiint g(x)(dx)^3$$
$$-f^3(x)\iint\iint g(x)(dx)^4 + \int(f^4(x)\int\int\int\int g(x)(dx)^4)dx \tag{7}$$

Since $f(x)$ is a polynomial of degree n and $g(x)$ can be integrated at least (n + 1) times, the procedure above will continue until the last stage where we now have:

$$\int f(x)g(x)dx = f(x)\int g(x)dx - f^1(x)\iint g(x)(dx)^2 + f^2(x)\iiint g(x)(dx)^3$$
$$-f^3(x)\iint\iint g(x)(dx)^4 + f^4(x)\iiint\iint g(x)(dx)^5 - f^5(x)\iiint\iiint g(x)(dx)^6$$
$$+ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots + (-1)^{n-1}[f^{n-1}(x)\int\int \dots(n)times\dots\int g(x)(dx)^n]$$
$$+ (-1)^n[\int(f^n(x)\int\int\dots(n)times\dots\int g(x)(dx)^n)dx] \tag{8}$$

$$\int f(x)g(x)dx = f(x)\int g(x)dx - f^1(x)\iint g(x)(dx)^2 + f^2(x)\iiint g(x)(dx)^3$$
$$-f^3(x)\iint\iint g(x)(dx)^4 + f^4(x)\iiint\iint g(x)(dx)^5 - f^5(x)\iiint\iiint g(x)(dx)^6$$
$$+ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots + (-1)^{n-1}[f^{n-1}(x)\int\int \dots(n)times\dots\int g(x)(dx)^n]$$
$$+ (-1)^n[f^n(x)\int\int\dots(n+1)times\dots\int g(x)(dx)^{n+1}] \tag{9}$$

Putting all these in a more compact form, we have:

$$\int f(x)g(x)dx = \sum_{j=0}^{n}(-1)^j f^j(x)\int g(x)^{[j+1]}\,dx \qquad (10)$$

where $f^j(x)$ is the j^{th} derivative of $f(x)$ and $\int g(x)^{[j+1]}\,dx$ represents the $(j + 1)^{th}$ integral of $g(x)$ with respect to x.

(10) above is the mathematical model for integrating product of two functions when the integrand involves a polynomial $f(x)$ of degree n and another function $g(x)$which can be integrated at least (n+1) times.

In order to apply the model to $\int f(x)g(x)dx$, we will find the derivatives of $f(x)$ up to the n^{th} derivative and the integrals of $g(x)$ up to the $(n + 1)^{th}$ integral and substitute all these into the formula in (10) above to get our final result directly.

3. Computational Examples

<u>Example 1</u>:

Let us consider $\int(x^4 + 5x^2 + 3x + 4)\, Sin3xdx$

$f(x) = x^4 + 5x^2 + 3x + 4$ and $g(x) = sin3x$

$f^1(x) = 4x^3 + 10x + 3$,

$f^2(x) = 12x^2 + 10,$

$f^3(x) = 24x$

and $f^4(x) = 24.$

From our notation:

$$\int g(x)^{[1]}\,dx = \int g(x)dx = \int sin3xdx = -\tfrac{1}{3}\, Cos3x.$$

$$\int g(x)^{[2]}\,dx = \iint g(x)(dx)^2 = \iint g(x)dxdx = -\tfrac{1}{9}\, Sin3x.$$

Similarly

$$\int g(x)^{[3]}\,dx = \tfrac{1}{27}\, Cos3x.$$

$$\int g(x)^{[4]}\,dx = \tfrac{1}{81}\, Sin3x$$

$$\int g(x)^{[5]}\,dx = -\tfrac{1}{243}\, Cos3x.$$

Substituting all these into the formula in (10) above we have our final result directly as:

$$\int(x^4 + 5x^2 + 3x + 4)\, Sin3xdx = -\tfrac{1}{3}(x^4 + 5x^2 + 3x + 4)Cos3x$$
$$+\tfrac{1}{9}(4x^3 + 10x + 3)Sin3x + \tfrac{1}{27}(12x^2 + 10)Cos3x - \tfrac{1}{81}(24x)Sin3x - \tfrac{24}{243}Cos3x + C.$$

where C is the constant of integration.

Using the well-known integration by parts method to solve the problem, we have:

$$\int(x^4+5x^2+3x+4)\,Sin3xdx = -\frac{1}{3}(x^4+5x^2+3x+4)Cos3x + \frac{1}{3}\int(4x^3+10x+3)Cos3xdx$$

.

$$= -\frac{1}{3}(x^4+5x^2+3x+4)Cos3x + \frac{1}{9}(4x^3+10x+3)Sin3x$$

$$-\frac{1}{9}\int(12x^2+10)\,\mathrm{Sin3xdx}$$

$$= -\frac{1}{3}(x^4+5x^2+3x+4)Cos3x + \frac{1}{9}(4x^3+10x+3)Sin3x + \frac{1}{27}(12x^2+10)Cos3x - \frac{1}{27}\int 24x\,Cos3x\,dx.$$

$$= -\frac{1}{3}(x^4+5x^2+3x+4)Cos3x + \frac{1}{9}(4x^3+10x+3)Sin3x + \frac{1}{27}(12x^2+10)\,Cos3x - \frac{1}{81}24x\,Sin3x + \frac{1}{81}\int 24\,Sin3xdx$$

.

$$\int(x^4+5x^2+3x+4)\,Sin3xdx = -\frac{1}{3}(x^4+5x^2+3x+4)Cos3x$$

$$+\frac{1}{9}(4x^3+10x+3)Sin3x + \frac{1}{27}(12x^2+10)Cos3x - \frac{1}{81}(24x)Sin3x - \frac{24}{243}Cos3x + C.$$

where C is the constant of integration.

This is the same with the result we got before.

Example 2:

Let us evaluate $\int(x^5+3x^4+2x)e^{2x}dx$.

$f(x) = x^5+3x^4+2x$ and $g(x) = e^{2x}$.

$f^1(x) = 5x^4+12x^3+2,$

$f^2(x) = 20x^3+36x^2,$

$f^3(x) = 60x^2+72x,$

$f^4(x) = 120x+72$

and $f^5(x) = 120.$

From our notation:

$$\int g(x)^{[1]}\,dx = \frac{1}{2}e^{2x}$$

$$\int g(x)^{[2]}\,dx = \frac{1}{4}e^{2x}$$

$\int g(x)^{[3]}\,dx = \frac{1}{8}e^{2x}$

$\int g(x)^{[4]}\,dx = \frac{1}{16}e^{2x}$

$\int g(x)^{[5]}\,dx = = \frac{1}{32}e^{2x}$

$\int g(x)^{[6]}\,dx = \frac{1}{64}e^{2x}$

Substituting all these into formula (10) above, we have our final result directly as:

$$\int(x^5 + 3x^4 + 2x)e^{2x}dx = [\tfrac{1}{2}(x^5 + 3x^4 + 2x) - \tfrac{1}{4}(5x^4 + 12x^3 + 2) + \tfrac{1}{8}(20x^3 + 36x^2) - \tfrac{1}{16}(60x^2 + 72x) + \tfrac{1}{32}(120x + 72) - \tfrac{15}{8}]\, e^{2x} + C.$$

where C is the constant of integration.

The integration by parts method, for this problem in example 2, gave us the same result after five routine applications.

Example 3:

Let us consider $\int x^2\, lnx dx$

$f(x) = x^2$ and $g(x) = lnx$.

$f^1(x) = 2x$ and $f^2(x) = 2$.

From our notation:

$\int g(x)^{[1]}\,dx = x(lnx - 1)$.

$\int g(x)^{[2]}\,dx = \frac{x^2}{2}(lnx - \frac{3}{2})$

$\int g(x)^{[3]}\,dx = \frac{x^3}{6}(lnx - \frac{11}{6})$

Putting all these in (10) above, we have our final result directly as:

$$\int x^2\, lnx dx = x^3(lnx - 1) - x^3\left(lnx - \frac{3}{2}\right) + \frac{x^3}{3}(lnx - \frac{11}{6}).$$

$$= \frac{x^3}{3}\left(lnx - \frac{1}{3}\right) + C.$$

where C is the constant of integration.

We also got the same result after two applications when we used the integration by parts method for the problem in example 3.

Example 4:

Let us consider $\int (x^5 + 3x^3 + 2x) Cosx\,dx$

$f(x) = x^5 + 3x^3 + 2x$ and $g(x) = Cosx$

$f^1(x) = 5x^4 + 9x^2 + 2,$

$f^2(x) = 20x^3 + 18x,$

$f^3(x) = 60x^2 + 18,$

$f^4(x) = 120x$

and $f^5(x) = 120.$

From our notation:

$\int g(x)^{[1]}\,dx = Sinx$

$\int g(x)^{[2]}\,dx = -Cosx$

$\int g(x)^{[3]}\,dx = -Sinx$

$\int g(x)^{[4]}\,dx = Cosx$

$\int g(x)^{[5]}\,dx = Sinx$

$\int g(x)^{[6]}\,dx - Cosx$

Putting all these in (10) above, we have our final result directly as:

$\int (x^5 + 3x^3 + 2x) cosx\,dx = (x^5 + 3x^3 + 2x) Sinx + (5x^4 + 9x^2 + 2) cosx -$
$(20x^3 + 18x) sinx - (60x^2 + 18) cosx + 120x Sinx + 120 Cosx + C.$

C is the constant of integration.

We got the same result for this example when we used the integration by parts method after five routine applications.

Conclusion

In this paper, a mathematical model for integrating product of two functions was presented for an integrand involving a polynomial of degree n. The advantage of the model is in the computational time that is saved. We don't need to go through the routine applications of integration by parts method each time we are integrating product of two functions. We only need to substitute the derivatives of $f(x)$ up to the nth derivative and the integrals of $g(x)$ up to the (n+1)th integral in the model. The mathematical model is found to be appropriate as it gives the same result with the integration by parts method.

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